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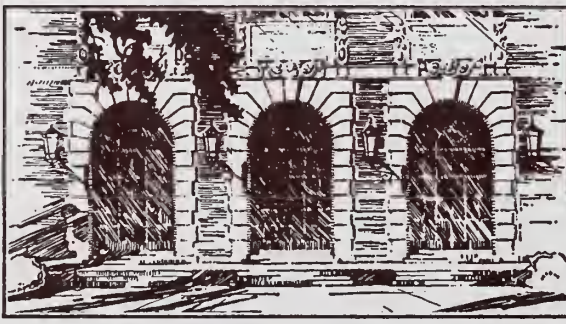
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A BOUND ON THE ERROR IN A GENERAL QUADRATURE
FORMULA WITH EQUIDISTANT ORDINATES

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Consider the equation

$$\int_{x_1}^{x_n} f(x) dx = \sum_{i=1}^n w_i f_i + \epsilon, \quad (1)$$

where the summation on the right represents an approximation to the integral and ϵ is the error in this approximation. Here $f_i \equiv f(x_i)$ and $x_{i+1} - x_i = h$ for $i = 1, 2, \dots, n-1$. We will find a bound on ϵ for an arbitrary quadrature formula, i.e., a bound in terms of the w_i which specify the particular formula under consideration. Equation (1) is a generalization of the well known Euler-Maclaurin formula¹ since that formula is merely (1) with the w_i chosen for the trapezoidal rule.²

Let us introduce the step function

$$\begin{aligned} \phi(x) &= x_i & x_i \leq x < x_{i+1} & \quad (i=1, \dots, n-1) \\ &= x_{n-1} & x = x_n & \end{aligned} \quad (2)$$

and then define

$$g(x) = x - \phi(x) \quad x_1 \leq x \leq x_n \quad (3)$$

Thus we can write

$$\begin{aligned} \sum_{i=1}^n w_i f_i &= \sum_{i=1}^n w_i f_i + \int_{x_1}^{x_n} [g(x) + \phi(x) - x] f'(x) dx \\ &= - \int_{x_1}^{x_n} x f'(x) dx + \sum_{i=1}^n w_i f_i + \int_{x_1}^{x_n} g(x) f'(x) dx \\ &\quad + \int_{x_1}^{x_n} \phi(x) f'(x) dx. \end{aligned}$$

Integrating the first integral by parts gives us

$$\begin{aligned} \sum_{i=1}^n w_i f_i &= \int_{x_1}^{x_n} f(x) dx - x f(x) \Big|_{x_1}^{x_n} + \sum_{i=1}^n w_i f_i + \int_{x_1}^{x_n} g(x) f'(x) dx \\ &\quad + \sum_{i=1}^n x_i \int_{x_i}^{x_{i+1}} f'(x) dx \end{aligned} \quad (4)$$

Comparing (4) with (1) we see that

$$\begin{aligned} -\epsilon &= x_1 f_1 - x_n f_n + \sum_{i=1}^n w_i f_i + \int_{x_1}^{x_n} g(x) f'(x) dx + \sum_{i=1}^{n-1} x_i (f_{i+1} - f_i) \\ &= w_1 f_1 + \sum_{n=2}^n (w_i - h) f_i + \int_{x_1}^{x_n} g(x) f'(x) dx \end{aligned} \quad (5)$$

Since $g(x)$ is a periodic function and satisfies the Dirichlet conditions we can write the Fourier series expansion

$$g(x) = h/2 - \sum_{m=1}^{\infty} (h/\pi m) \cos \pi m \sin [(2\pi m/h)(x - x_1 - h/2)] \quad (6)$$

for the range $x_1 \leq x \leq x_n$. Hence, if $\bar{x} = x - x_1 - h/2$, the last term in (5) can be written

$$\begin{aligned} \int_{x_1}^{x_n} g(x) f'(x) dx &= (h/2) \int_{x_1}^{x_n} f'(x) dx \\ &\quad - \int_{x_1}^{x_n} f'(x) \left[\sum_{m=1}^{\infty} (h/\pi m) \cos \pi m \sin(2\pi m \bar{x}/h) \right] dx \\ &= (h/2)(f_n - f_1) - \sum_{m=1}^{\infty} (-1)^m (h/\pi m) \int_{x_1}^{x_n} f'(x) \sin(2\pi m \bar{x}/h) dx \end{aligned}$$

if we assume that $f'(x)$ is of bounded variation.³

In order to use (7) we need the relation, obtained by integrating by parts t times,

$$\begin{aligned} \int_{x_1}^{x_n} f'(x) \sin(2\pi m \bar{x}/h) dx &= \sum_{i=1}^t (h/2\pi m)^i A_i [f_n^i S_i(x_n) - f_1^i S_i(x_1)] \\ &\quad - A_t (h/2\pi m)^t \int_{x_1}^{x_n} f^{t+1}(x) S_t(x) dx \end{aligned} \quad (8)$$

where

$$A_i = (-1)^{(i+2)(i+1)/2}$$

and

$$\begin{aligned} S_i(x) &= \sin(2\pi m \bar{x}/h) \quad \text{if } i \text{ is even} \\ &= \cos(2\pi m \bar{x}/h) \quad \text{if } i \text{ is odd.} \end{aligned}$$

Substituting (8) into (7) gives us

$$\int_{x_1}^{x_n} g(x) f'(x) dx = (h/2)(f_n - f_1) - \sum_{m=1}^{\infty} (-1)^m (h/\pi m) \left[\sum_{i=1}^t A_i (h/2\pi m)^i \cdot \{f_n^i S_i(x_n) - f_1^i S_i(x_1)\} - A_t (h/2\pi m)^t \int_{x_1}^{x_n} f^{t+1} S_t(x) dx \right]$$

In order to write the infinite series in (9) as the sum of two series we must show that it is absolutely convergent. To do this we write the m^{th} term in the form

$$m^{-2} [(-1)^m (h/\pi) \left\{ \sum_{i=1}^t A_i (h/2\pi)^i m^{1-i} (f_n^i S_i(x_n) - f_1^i S_i(x_1)) - A_t (h/2\pi)^t m^{1-t} \int_{x_1}^{x_n} f^{t+1} S_t(x) dx \right\}] \quad (10)$$

If the first $t+1$ derivatives of $f(x)$ are bounded then the term in brackets in (10) is bounded. Thus, since $\sum_{m=1}^{\infty} M m^{-2}$ converges (here M is a bound on the term in brackets) the series in (9) is absolutely convergent by the comparison test. Hence

$$\int_{x_1}^{x_n} g(x) f'(x) dx = (h/2)(f_n - f_1) - \sum_{m=1}^{\infty} (-1)^m (h/\pi m) \left[\sum_{i=1}^t A_i (h/2\pi m)^i \cdot \{f_n^i S_i(x_n) - f_1^i S_i(x_1)\} + \sum_{m=1}^{\infty} (-1)^m 2A_t (h/2\pi m)^{t+1} \int_{x_1}^{x_n} f^{t+1} S_t(x) dx \right], \quad (11)$$

and substituting into (5) we get

$$\begin{aligned} -\epsilon &= (w_1 - h/2) f_1 + (w_n - h/2) f_n + \sum_{i=2}^{n-1} (w_i - h) f_i \\ &- \sum_{m=1}^{\infty} (-1)^m \sum_{i=1}^t 2A_i (h/2\pi m)^{i+1} \{f_n^i S_i(x_n) - f_1^i S_i(x_1)\} \\ &+ \sum_{m=1}^{\infty} (-1)^m 2A_t (h/2\pi m)^{t+1} \int_{x_1}^{x_n} f^{t+1} S_t(x) dx \end{aligned} \quad (12)$$

Examine the last term in (12). Since $|S_t(x)| \leq 1$ we know that if $|f^{t+1}|$ is bounded by K

$$\left| \int_{x_1}^{x_n} f^{t+1} S_t(x) dx \right| \leq \int_{x_1}^{x_n} |f^{t+1}| dx \leq K (n-1) h \quad (13)$$

Thus

$$\left| \sum_{m=1}^{\infty} (-1)^m 2A_t(h/2\pi m)^{t+1} \int_{x_1}^{x_n} f^{t+1} S_t(x) dx \right| \leq K(n-1) h^{t+2} / 2^t \pi^{t+1} \sum_{m=1}^{\infty} m^{-(t+1)} \quad (14)$$

and if we assume $t \geq 1$ we have

$$\sum_{m=1}^{\infty} m^{-(1+t)} \leq \pi^2/6. \quad (15)$$

Hence with this restriction on t the term is bounded by

$$(K/3) (n-1) h^{t+2} \pi^{1-t} 2^{-(t+1)} \quad (16)$$

The other infinite series in (12) is absolutely convergent and so it can be written as the sum of series. To do this we write

$$\begin{aligned} & \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{i=1}^t 2A_i(h/2\pi m)^{i+1} \left\{ f_n^i S_i(x_n) - f_1^i S_i(x_1) \right\} \\ &= \sum_{m=1}^{\infty} (-1)^{m+1} [-(h^2/2\pi^2 m^2) \{ f_n' \cos(2n-3)\pi m - f_1' \cos \pi m \} \\ &+ (h^3/4\pi^3 m^3) \{ f_n'' \sin(2n-3)\pi m + f_1'' \sin \pi m \} \\ &+ (h^4/8\pi^4 m^4) \{ f_n^{(3)} \cos(2n-3)\pi m - f_1^{(3)} \cos \pi m \} + \dots \\ &+ 2A_{t-1}(h/2\pi m)^t \left\{ f_n^{t-1} S_{t-1}(x_n) - f_1^{t-1} S_{t-1}(x_1) \right\} \\ &+ 2A_t(h/2\pi m)^{t+1} \left\{ f_n^t S_t(x_n) - f_1^t S_t(x_1) \right\} \quad 1. \end{aligned} \quad (17)$$

From the definitions of $S_i(x)$ it follows that one of the last two terms will vanish since, for either $i = t$ or $i = t-1$, i is even and

$$S_i(x_n) = \sin(2n-3)\pi m = 0 \quad (18)$$

$$S_i(x_1) = \sin \pi m = 0.$$

We denote by T the odd integer (t or $t-1$) which gives

$$\begin{aligned} S_T(x_n) &= \cos(2n-3)\pi m = (-1)^{m(2n-3)} \\ S_T(x_1) &= \cos \pi m = (-1)^m. \end{aligned} \quad (19)$$

Then the right side of (17) becomes

$$\begin{aligned}
&= (h^2/2\pi^2) [f'_n \sum_{m=1}^{\infty} m^{-2} (-1)^{2m(n-1)} + f'_1 \sum_{m=1}^{\infty} m^{-2} (-1)^{2m-1}] \\
&\quad + (h^4/8\pi^4) [f_n^3 \sum_{m=1}^{\infty} m^{-4} (-1)^{2m(n-1)+1} - f_1^3 \sum_{m=1}^{\infty} m^{-4} (-1)^{2m+1}] \\
&\quad + \dots \\
&\quad + 2A_T (h/2\pi)^{T+1} [f_n^T \sum_{m=1}^{\infty} m^{-(T+1)} (-1)^{2m(n-1)+1} - f_1^T \sum_{m=1}^{\infty} m^{-(T+1)} (-1)^{2m+1}] \\
&= (h^2/2\pi^2) (f'_n - f'_1) \sum_{m=1}^{\infty} m^{-2} - (h^4/8\pi^4) (f_n^3 - f_1^3) \sum_{m=1}^{\infty} m^{-4} \\
&\quad + \dots - 2A_T (h/2\pi)^{T+1} (f_n^T - f_1^T) \sum_{m=1}^{\infty} m^{-(T+1)}. \quad (20)
\end{aligned}$$

Since $\sum_{m=1}^{\infty} m^{-2p} = (-1)^{p-1} B_{2p} (2\pi)^{2p}/2(2p)!$ for fixed integers p (here B_{2p} are Bernoulli's numbers), we have $\sum_{m=1}^{\infty} m^{-2} = \pi^2/6$,

$$\sum_{m=1}^{\infty} m^{-4} = \pi^4/90, \text{ and } \sum_{m=1}^{\infty} m^{-(T+1)} = (-1)^{(T-1)/2} B_{T+1} (2\pi)^{T+1}/2(T+1)!.$$

Thus the right side of (20) becomes

$$\begin{aligned}
&= (h^2/12) (f'_n - f'_1) - (h^4/720) (f_n^3 - f_1^3) + \dots \\
&\quad + (-1)^{(T+1)^2/2 + T} (h^{T+1} B_{T+1}/(T+1)!) (f_n^T - f_1^T). \quad (21)
\end{aligned}$$

Using (21), and (12) in (1) we get

$$\begin{aligned}
\int_{x_1}^{x_n} f(x) dx &= \sum_{i=1}^n w_i f_i - [(w_1 - h/2) f_1 + (w_n - h/2) f_n + \sum_{i=2}^{n-1} (w_i - h) f_i \\
&\quad + (h^2/12)(f'_n - f'_1) - (h^4/720)(f_n^3 - f_1^3) + \dots \\
&\quad + (-1)^{(T+1)^2/2 + T} (h^{T+1} B_{T+1}/(T+1)!) (f_n^T - f_1^T) \\
&\quad + 2 \sum_{m=1}^{\infty} (-1)^m A_t (h/2\pi m)^{t+1} \int_{x_1}^{x_n} f^{t+1} S_t(x) dx] \quad (22)
\end{aligned}$$

which demonstrates² the relation with the Euler-Maclaurin formula.

If we use (21) and (16) with (12) we see that, under the assumptions we have made,

$$\begin{aligned}
 |\epsilon| \leq & (w_1 - h/2) f_1 + (w_n - h/2) f_n + \sum_{i=2}^{n-1} (w_i - h) f_i \\
 & + (h^2/12) (f'_n - f'_1) - (h^4/720) (f_n^3 - f_1^3) + \dots \\
 & + (-1)^{(T+1)^2/2 + T} (h^{T+1} B_{T+1} / (T+1)!) (f_n^T - f_1^T) | \\
 & + (K/3) (n-1) h^{t+2} \pi^{1-t} 2^{-(t+1)}
 \end{aligned} \tag{23}$$

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